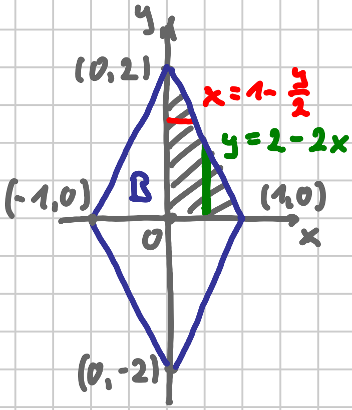
SS 2015 Analysis 4 LB Hausaufgaben Blatt 9

Notiztitel 06.05.2011

H14: $\iint\limits_B \underbrace{x^4+y^4}_{=:f(x,y)}\,dx\,dy = V$



a) B = Raute, Mittelpunkt (0,0)

Ecken $(\pm 1, 0)$; $(0, \pm 2)$

$f(x,y) = x^4 + y^4$ und Raute B

sind achsensymmetrisch

wegen $f(\pm x, \pm y) = f(x,y) \Rightarrow V = 4 \cdot V_\Delta$ (schraffierter Bereich)

ⓐ erst über x dann über y

$$V = 4 \cdot \int_{y=0}^{2} \left(\int_{x=0}^{1-\frac{y}{2}} x^4 + y^4 \, dx \right) dy = 4 \cdot \int_{y=0}^{2} \left[\tfrac{1}{5} x^5 + y^4 \cdot x \right]_{x=0}^{x=1-\frac{y}{2}} dy =$$

$$= 4 \cdot \int_{y=0}^{2} \tfrac{1}{5} \cdot \tfrac{1}{32} (2-y)^5 + \underbrace{y^4 \left(1 - \tfrac{y}{2}\right)}_{= y^4 - \frac{1}{2} y^5} dy =$$

$$= 4 \cdot \left[\tfrac{1}{5} \cdot \tfrac{1}{32} \cdot \tfrac{-1}{6} (2-y)^6 + \tfrac{1}{5} y^5 - \tfrac{1}{12} y^6 \right]_{y=0}^{2} =$$

$$= 4 \cdot \left[\tfrac{1}{5} \cdot 32 - \tfrac{1}{12} \cdot 64 + \tfrac{1}{5} \cdot \tfrac{1}{32} \cdot \tfrac{1}{6} \cdot 64 \right] = 4 \cdot \left[\tfrac{32}{5} - \tfrac{16}{3} + \tfrac{1}{15} \right] = \underline{\underline{\tfrac{68}{15}}}$$

ⓑ erst über y dann über x

$$V = 4 \cdot \int_{x=0}^{1} \left(\int_{y=0}^{2-2x} x^4 + y^4 \, dy \right) dx = 4 \cdot \int_{x=0}^{1} \left[x^4 \cdot y + \tfrac{1}{5} y^5 \right]_{y=0}^{2-2x} dx =$$

$$= 4 \cdot \int_{x=0}^{1} \underbrace{x^4 (2-2x)}_{2x^4 - 2x^5} + \tfrac{1}{5} \cdot 32 (1-x)^5 \, dx =$$

$$= 4 \cdot \left[\tfrac{2}{5} x^5 - \tfrac{2}{6} x^6 - \tfrac{1}{5} \cdot 32 \cdot \tfrac{1}{6} (1-x)^6 \right]_{x=0}^{1} =$$

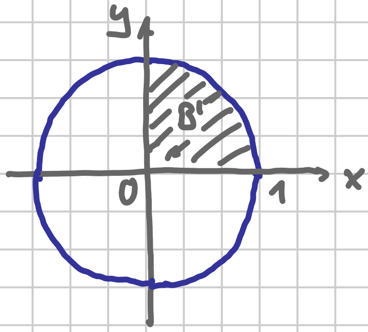
$$= 4 \cdot \left[\tfrac{2}{5} - \tfrac{2}{6} + \tfrac{1}{5} \cdot 32 \cdot \tfrac{1}{6} \right] = 4 \cdot \frac{12 - 10 + 32}{30} = \underline{\underline{\tfrac{68}{15}}}$$

Bem. Ohne Ausnutzung der Symmetrie wäre die Raute B in 2 Normalber. bei ⓐ in △▽ und bei ⓑ in ◁▷ zu zerlegen!

b) B = Kreis um $M=(0,0)$ mit Radius 1

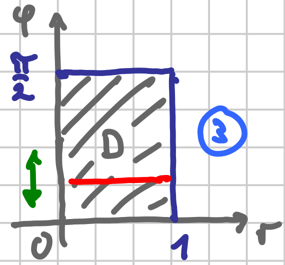
$f(x,y)=x^4+y^4$ und Kreis B sind achsensymmetrisch (vgl. oben) $\Rightarrow$

$V = 4\cdot V_D$ (Viertelkreis B')



Zweckmäßig sind Polarkoord.

① $\begin{array}{l} x = r\cdot\cos\varphi \\ y = r\cdot\sin\varphi \end{array} \Rightarrow dx\,dy = r\,dr\,d\varphi$ ②



$f(x,y) = f(r\cos\varphi, r\sin\varphi) = r^4(\cos^4\varphi+\sin^4\varphi)$

$$V = 4\cdot\iint_{B'} f(\underbrace{x,y}_{①})\,\underbrace{dx\,dy}_{} = 4\cdot\iint_{D\,③} f(\underbrace{r\cos\varphi, r\sin\varphi}_{①})\,\underbrace{r\,dr\,d\varphi}_{②} =$$

$$= 4\cdot\int_{\varphi=0}^{\pi/2}\left(\int_{r=0}^{1} r^5(\cos^4\varphi+\sin^4\varphi)\,dr\right)d\varphi = 4\cdot\int_{\varphi=0}^{\pi/2} \frac{1}{6}r^6(\cos^4\varphi+\sin^4\varphi)\Big|_{r=0}^{1}\,d\varphi =$$

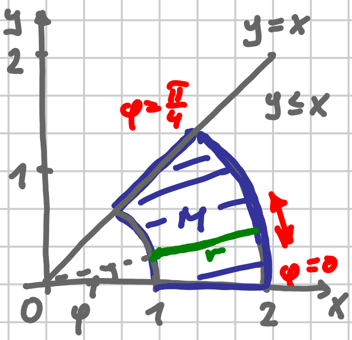
quad. Ergänzung

$$= \frac{4}{6}\cdot\int_{\varphi=0}^{\pi/2} \underbrace{(\cos^2\varphi+\sin^2\varphi)}_{=1}{}^2 - 2\underbrace{\cos^2\varphi\sin^2\varphi}_{=\frac{1}{4}\sin^2(2\varphi)}\,d\varphi =$$

$$= \frac{1}{3}\int_{\varphi=0}^{\pi/2} 2-\sin^2(2\varphi)\,d\varphi = \frac{1}{3}\cdot\left[2\varphi-\frac{1}{2}\varphi+\frac{1}{8}\sin 4\varphi\right]_{\varphi=0}^{\pi/2} = \underline{\underline{\frac{\pi}{4}}}$$

H15 $M := \{(x,y)\in\mathbb{R}^2 \mid 4\geq x^2+y^2\geq 1,\ x\geq y\geq 0\}$

wird durch Polarkoord.: $\begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \end{array}$ zu



③ $D := \{(r,\varphi) \mid 1\leq r\leq 2,\ 0\leq\varphi\leq\frac{\pi}{4}\}$

① $f(x,y) = \frac{y}{x^4} = \underbrace{\frac{r\sin\varphi}{r^4\cos^4\varphi}}_{f(r\cos\varphi,\ r\sin\varphi)}$; ② $dx\,dy = r\,dr\,d\varphi$

$$\iint_M f(\underbrace{x,y}_{})\,\underbrace{dx\,dy}_{} = \iint_{D\,③} f(r\cos\varphi, r\sin\varphi)\cdot \overbrace{r\,dr\,d\varphi}^{②} = \int_{\varphi=0}^{\pi/4}\left(\int_{r=1}^{2}\frac{r\sin\varphi}{r^4\cos^4\varphi}\,r\,dr\right)d\varphi =$$

$$= \int_{\varphi=0}^{\pi/4}\frac{\sin\varphi}{\cos^4\varphi}\cdot\left[-\frac{1}{r}\right]_{r=1}^{2}d\varphi = \int_{\varphi=0}^{\pi/4}\frac{1}{2}\cdot\frac{\sin\varphi}{\cos^4\varphi}\,d\varphi = \frac{1}{2}\int_{u=1}^{1/\sqrt{2}} -\frac{1}{u^4}\,du = \frac{1}{2}\cdot\frac{1}{3}u^{-3}\Big|_{u=1}^{1/\sqrt{2}} =$$

Subst. ① $u = \cos\varphi$ ② $du = -\sin\varphi\,d\varphi$ ③ $\varphi_0 = 0 \Rightarrow u_0 = \cos 0 = 1$; $\varphi_1 = \pi/4 \Rightarrow u_1 = \cos\pi/4 = \frac{1}{2}\sqrt{2} = \frac{1}{\sqrt{2}}$

$$= \underline{\underline{\frac{1}{6}(2\sqrt{2}-1)}}$$

H 14 b alternativ direkt:

$$\int_{x=0}^{1}\left(\int_{y=0}^{\sqrt{1-x^2}} x^4+y^4\,dy\right)dx = \int_{x=0}^{1} x^4\cdot y+\frac{1}{5}y^5\Big|_{y=0}^{\sqrt{1-x^2}}dx = \int_{x=0}^{1} x^4\sqrt{1-x^2}+\frac{1}{5}\underbrace{(1-x^2)^2}_{1-2x^2+x^4}\cdot\sqrt{1-x^2}\,dx$$

$$=\frac{1}{5}\int_{x=0}^{1} \sqrt{1-x^2}-2x^2\sqrt{1-x^2}+6x^4\sqrt{1-x^2}\,dx=$$

$$=\frac{1}{5}\Big[\underset{\frac{4}{8}}{\frac{1}{2}}\cdot\left(\arcsin x + x\sqrt{1-x^2}\right) - \underset{\frac{2}{8}}{\frac{1}{4}}\left(\arcsin x - x\sqrt{1-x^2}+2x^3\sqrt{1-x^2}\right) +$$

$$+\frac{1}{8}\left(3\arcsin x - 3x\sqrt{1-x^2}-2x^3\sqrt{1-x^2}+8x^5\sqrt{1-x^2}\right)\Big]=$$

$$=\frac{1}{40}\Big[5\cdot\arcsin x+3x\sqrt{1-x^2}-6x^3\sqrt{1-x^2}+8x^5\sqrt{1-x^2}\Big]\Big|_{x=0}^{1}=$$

$$=\frac{1}{8}\cdot(\arcsin 1-\arcsin 0)=\frac{\pi}{16} \Rightarrow V=4\cdot\frac{\pi}{16}=\frac{\pi}{4}$$ ✓