SS 2015 Analysis 4 LB Übungen Blatt 8

Notiztitel 16.05.2013

T23 $f(x,y) = (2-xy)\,xy\,e^{-xy}$ (uneigentliche Integrale)

① $$\int_{y=0}^{\infty} f(x,y)\,dy = \lim_{b\to\infty} \int_{y=0}^{b} (2-xy)\,xy\,e^{-xy}\,dy = \lim_{b\to\infty} \left(xy^2 e^{-xy}\right)\Big|_{y=0}^{y=b}$$

(Stammfkt in y bei const. x)

$$= \lim_{b\to\infty} x\,b^2 \cdot e^{-xb} = \underline{0}$$ für alle $x \geq 0$ insbes. $x \in [0,1]$

$$\Rightarrow \int_{x=0}^{1} \left(\int_{y=0}^{\infty} f(x,y)\,dy \right) dx = \int_{x=0}^{1} 0\,dx = \underline{0}$$

(Stammfkt in x bei const y)

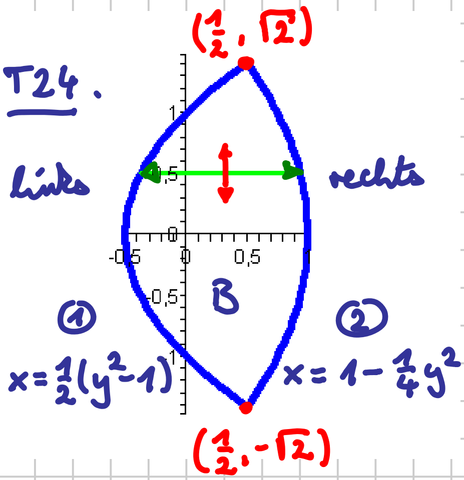
② $$\int_{x=0}^{1} f(x,y)\,dx = \int_{x=0}^{1} (2-xy)\,xy\,e^{-xy}\,dx = \left(x^2 y\, e^{-xy}\right)\Big|_{x=0}^{x=1} = y \cdot e^{-y}$$

$$\Rightarrow \int_{y=0}^{\infty} \left(\int_{x=0}^{1} f(x,y)\,dx \right) dy = \int_{y=0}^{\infty} y\,e^{-y}\,dy = \lim_{b\to\infty} \int_{y=0}^{b} \underbrace{y}_{u}\,\underbrace{e^{-y}}_{v'}\,dy =$$

$$= \lim_{b\to\infty} \Big[\underbrace{-y e^{-y}}_{u\cdot v}\Big|_{y=0}^{y=b} - \int_0^b \underbrace{-e^{-y}}_{u'=1,\ v}\,dy \Big] = \lim_{b\to\infty} \left(-b e^{-b} - [e^{-y}]\Big|_{y=0}^{y=b} \right) =$$

$$= \lim_{b\to\infty} \left(-b e^{-b} - e^{-b} + 1 \right) = \underline{1 \neq 0}$$ nach ①

T24.



Schnittpunkte der Parabeln ① ∧ ②:

$$\frac12 (y^2 - 1) = 1 - \frac14 y^2 \Leftrightarrow \frac34 y^2 = \frac32$$

$$\Leftrightarrow y^2 = 2 \Leftrightarrow \underline{y = \pm\sqrt2} \overset{①}{\underset{②}{\Rightarrow}} \underline{x = \frac12}$$

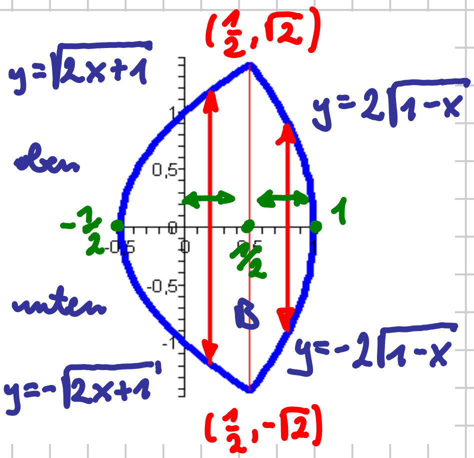
1. Weg: „Erst über x dann über y"

$$\iint_B dF = \int_{y=-\sqrt2}^{\sqrt2} \left(\int_{x=\frac12(y^2-1)}^{1-\frac14 y^2} 1 \cdot dx \right) dy =$$

$$= \int_{-\sqrt2}^{\sqrt2} \left[1 - \frac14 y^2 - \frac12 (y^2 - 1) \right] dy = 2 \cdot \int_0^{\sqrt2} \frac32 - \frac34 y^2\,dy = 2 \cdot \left[\frac32 y - \frac14 y^3 \right]_0^{\sqrt2} =$$

Symmetrie!

$$= 2\left(\frac32 \sqrt2 - \frac12 \sqrt2 \right) = \underline{2\sqrt2}$$



<u>2. Weg</u>: „Erst über y dann über x“
aufgeteilt in 2 Normalbereiche

$$\iint_B dF = \int_{x=-1/2}^{1/2}\left(\int_{y=-\sqrt{2x+1}}^{\sqrt{2x+1}} 1\,dy\right)dx + \int_{x=1/2}^{1}\left(\int_{y=-2\sqrt{1-x}}^{2\sqrt{1-x}} 1\,dy\right)dx =$$

$$= \int_{-1/2}^{1/2} 2\sqrt{2x+1}\,dx + \int_{1/2}^{1} 4\sqrt{1-x}\,dx =$$

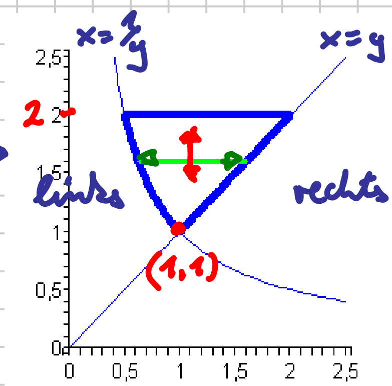
$$= \frac{2}{3}(2x+1)^{3/2}\Big|_{-1/2}^{1/2} + \left(-\frac{8}{3}(1-x)^{3/2}\right)\Big|_{1/2}^{1} =$$

$$= \frac{2}{3}\,2^{3/2} + \frac{8}{3}\cdot\left(\frac12\right)^{3/2} = \frac43\sqrt2 + \frac23\sqrt2 = \underline{\underline{2\sqrt2}}$$

<u>T25.</u>

$$\int_{y=1}^{2}\left(\int_{x=1/y}^{y}\frac{y^2}{x^2}\,dx\right)dy = \int_{y=1}^{2}\left(-\frac{y^2}{x}\Big|_{x=1/y}^{y}\right)dy =$$

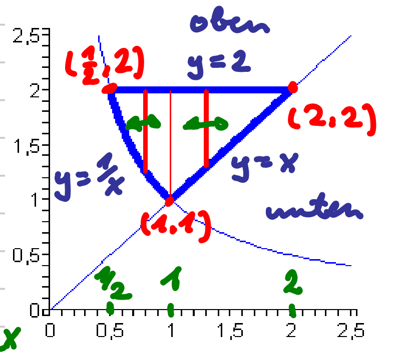
Erst längs x von $x=\frac1y$ bis $x=y$
dann längs y von $y=1$ bis $y=2$



$$= \int_{y=1}^{2}(-y+y^3)\,dy = -\frac12 y^2 + \frac14 y^4\Big|_{y=1}^{2} =$$

$$= -2+4+\frac12-\frac14 = \underline{\underline{\frac94}}$$

<u>2. Weg</u> durch Vertauschen der Integrationsreihenfolge nach Fubini.
Erst über y dann über x aufgeteilt in 2 Normalbereiche:



$$\iint_B \frac{y^2}{x^2}\,dF = \int_{x=1/2}^{1}\left(\int_{y=1/x}^{2}\frac{y^2}{x^2}\,dy\right)dx + \int_{x=1}^{2}\left(\int_{y=x}^{2}\frac{y^2}{x^2}\,dy\right)dx$$

$$= \int_{x=1/2}^{1}\frac13\,\frac{y^3}{x^2}\Big|_{y=1/x}^{2}dx + \int_{x=1}^{2}\frac13\,\frac{y^3}{x^2}\Big|_{y=x}^{2}dx =$$

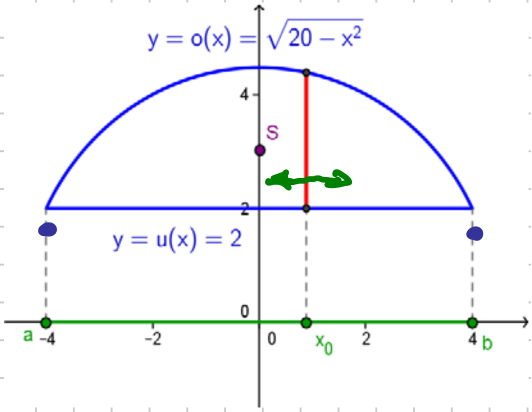
$$= \int_{x=1/2}^{1}\frac13\left(\frac{8}{x^2}-\frac{1}{x^5}\right)dx + \int_{x=1}^{2}\frac13\left(\frac{8}{x^2}-x\right)dx =$$

$$= \frac13\left[\frac{-8}{x}+\frac14\cdot\frac{1}{x^4}\right]_{x=1/2}^{1} + \frac13\left[\frac{-8}{x}-\frac12x^2\right]_{x=1}^{2} =$$

$$= \frac13\left[-\cancel{8}+\frac14+16-4\right]+\frac13\left[-4-2+\cancel{8}+\frac12\right] = \dots = \underline{\underline{\frac94}}$$

T26 $B := \{(x,y) \mid x^2+y^2 \le 20,\ y \ge 2\}$

Schnittpunkte der Randlinien



$y=2$ in $x^2+y^2=20^2 \Rightarrow x^2=16 \Rightarrow x=\pm 4$

$\Rightarrow$ Schnittpunkte $(-4,2)$ und $(4,2)$

Integration zuerst über y von

$y=u(x)=2$ bis $y=o(x)=\sqrt{20-x^2}$, dann über x von a nach b

Fläche von B: $F=\iint\limits_B 1\,dF = \int_{-4}^{4}\left(\int_{2}^{\sqrt{20-x^2}} 1\,dy\right)dx = \int_{-4}^{4} \sqrt{20-x^2}-2\,dx =$

$= \frac{1}{2}\left(x\sqrt{20-x^2}+20\arcsin\frac{x}{\sqrt{20}}\right)-2x\Big|_{-4}^{4} =$

$= 20\arcsin\frac{2}{\sqrt{5}} - 8 \approx 14{,}14297$

$y_s = \frac{1}{F}\cdot\iint\limits_B y\,dF = \frac{1}{F}\cdot\int_{-4}^{4}\left(\int_{2}^{\sqrt{20-x^2}} y\,dy\right)dx = \int_{-4}^{4} \frac{1}{2}(20-x^2)-\frac{1}{2}\cdot 2^2\,dx =$

$= \frac{1}{F}\left(8x-\frac{1}{6}x^3\Big|_{-4}^{4}\right) = \frac{1}{F}\left(64-\frac{1}{3}\cdot 64\right) = \frac{2}{3}\cdot\frac{64}{F} \approx 3{,}0168$

$x_s = 0$ da B symmetrisch zur y-Achse!

$\left(x_s = \frac{1}{F}\iint\limits_B x\,dF = \frac{1}{F}\cdot\int_{-4}^{4}\left(\int_{2}^{\sqrt{20-x^2}} x\,dy\right)dx = \int_{-4}^{4} \underbrace{x\cdot\sqrt{20-x^2}-2x}_{\text{punktsymm.}}\,dx = 0\right)$

$\Rightarrow S \approx (0,\ 3{,}0168)$