SS 2015 Analysis 4 LB Übungen Blatt 4

Notiztitel 26.05.2011

T12

$$\operatorname{grad}(f\cdot g) = \begin{pmatrix} (f\cdot g)_{x_1} \\ \vdots \\ (f\cdot g)_{x_n} \end{pmatrix} \underset{\text{Produktregel}}{=} \begin{pmatrix} f_{x_1}\cdot g + f\cdot g_{x_1} \\ \vdots \\ f_{x_n}\cdot g + f\cdot g_{x_n} \end{pmatrix} =$$

$$\underbrace{h(\underline{x})}_{\underline{x}\in\mathbb{R}^n} = \underbrace{f(\underline{x})}_{\in\mathbb{R}}\cdot\underbrace{g(\underline{x})}_{\in\mathbb{R}} \in \mathbb{R}$$

$$= \begin{pmatrix} f_{x_1} \\ \vdots \\ f_{x_n} \end{pmatrix}\cdot g + f\cdot\begin{pmatrix} g_{x_1} \\ \vdots \\ g_{x_n} \end{pmatrix} = (\operatorname{grad} f)\cdot g + f\cdot\operatorname{grad} g =$$

$$= g\cdot\operatorname{grad} f + f\cdot\operatorname{grad} g$$

T13

$$f(x,y) = x^2y + xy^3 + x \quad\Rightarrow\quad \begin{aligned} f_x(x,y) &= 2xy + y^3 + 1 \\ f_y(x,y) &= x^2 + 3xy^2 \end{aligned}$$

a) $$\nabla f(1,2) = \operatorname{grad} f(1,2) = \begin{pmatrix} f_x(1,2) \\ f_y(1,2) \end{pmatrix} = \begin{pmatrix} 13 \\ 13 \end{pmatrix}$$

Richtungsableitung $\underline{a} \parallel \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ mit $|\underline{a}| = 1$ d.h. $\underline{a} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$:

$$\frac{\partial}{\partial \underline{a}} f(1,2) = \langle \operatorname{grad} f(1,2), \underline{a} \rangle = \left\langle \begin{pmatrix} 13 \\ 13 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle = 0$$

(d.h. Der Schnitt durch $(1,2)$ in Richtung $(1,-1)$ hat horizontale Tangente)

b) Mit den Abkürzungen $s := \sin t$, $c := \cos t$ gilt:

$$z(t) = f(\underbrace{1+s}_{x(t)}, \underbrace{1-c}_{y(t)}) = (1+s)^2(1-c) + (1+s)(1-c)^3 + (1+s)$$

1. Weg direkt:

$$z'(t) = 2\cdot(1+s)\cdot c\cdot(1-c) + (1+s)^2 s + c(1-c)^3 + (1+s)3(1-c)^2\cdot s + c$$

2. Weg Kettenregel mit $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 1+\sin t \\ 1-\cos t \end{pmatrix} \Rightarrow \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

Kreis um $(1,1)$ vom Radius 1

$$z'(t) = \langle \operatorname{grad} f(x(t), y(t)), (\dot{x}(t), \dot{y}(t))^T \rangle =$$

$$= \left\langle \begin{pmatrix} 2(1+s)(1-c) + (1-c)^3 + 1 \\ (1+s)^2 + 3(1+s)(1-c)^2 \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix} \right\rangle =$$

$$= 2c(1+s)(1-c) + c(1-c)^3 + c + s(1+s)^2 + 3s(1+s)(1-c)^2$$

c) $\tau: \begin{pmatrix} f_x(1,2) \\ f_y(1,2) \\ -1 \end{pmatrix} \cdot \left[\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ f(1,2) \end{pmatrix} \right] = 0 \iff \begin{pmatrix} 13 \\ 13 \\ -1 \end{pmatrix} \left[\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 11 \end{pmatrix} \right] = 0$

$\iff 13x + 13y - z - 13 - 26 + 11 = 0 \iff \underline{13x + 13y - z = 28}$

d) Horizontale Tangentenebene ($z = \text{const}$) $\iff \operatorname{grad} f = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\iff \left\{ \begin{matrix} f_x = 0 \\ f_y = 0 \end{matrix} \right\} \iff \begin{cases} 2xy + y^3 + 1 = 0 \\ x^2 + 3xy^2 = 0 \end{cases} \iff \left\{ \begin{matrix} x = 0 & \text{und} & y^3 + 1 = 0 \\ \text{oder} & & \\ x = -3y^2 & \text{und} & -5y^3 + 1 = 0 \end{matrix} \right\}$

$\Rightarrow (x = 0 \land y = -1)$ oder $\left(y = \frac{1}{\sqrt[3]{5}} \land x = \frac{-3}{\sqrt[3]{25}}\right)$

T14. $u(x,t) = -2\cosh^{-2}(x-4t)$

$u_t = 4\cosh^{-3}(x-4t) \cdot \sinh(x-4t) \cdot (-4) = -16 \frac{\sinh(x-4t)}{\cosh^3(x-4t)}$

$u_x = 4\cosh^{-3}(x-4t) \cdot \sinh(x-4t) = 4 \frac{\sinh(x-4t)}{\cosh^3(x-4t)}$

$u_{xx} = 4 \cdot \frac{\cosh^4(x-4t) - \sinh^2(x-4t) \cdot 3\cosh^2(x-4t)}{\cosh^6(x-4t)} =$

$= 4 \cdot \frac{\cosh^2(x-4t) - 3\sinh^2(x-4t)}{\cosh^4(x-4t)} \underset{\cosh^2(x) - \sinh^2(x) = 1}{=} 4 \frac{-2\cosh^2(x-4t) + 3}{\cosh^4(x-4t)}$

$= \frac{-8}{\cosh^2(x-4t)} + \frac{12}{\cosh^4(x-4t)}$

$u_{xxx} = +16 \frac{\sinh(x-4t)}{\cosh^3(x-4t)} - 48 \frac{\sinh(x-4t)}{\cosh^5(x-4t)}$

$\underline{u u_x - u_{xxx}} = -48 \frac{\sinh(x-4t)}{\cosh^5(x-4t)} - 16 \frac{\sinh(x-4t)}{\cosh^3(x-4t)} + 48 \frac{\sinh(x-4t)}{\cosh^5(x-4t)} = \underline{u_t}$

qed.

T15 Tangentenebene τ an $H: \underbrace{x^2 + y^2 - z^2}_{=: F(x,y,z)} = 1$ [$F(x,y,z) = c = \text{const}$]

im Punkt $(1,1,1) \in H$ mit $\operatorname{grad} F = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ -2z \end{pmatrix}$

($F(1,1,1) = 1$ ✓)

$F_x(x_0 y_0 z_0)(x - x_0) + F_y(x_0 y_0 z_0)(y - y_0) + F_z(x_0 y_0 z_0)(z - z_0) = 0$

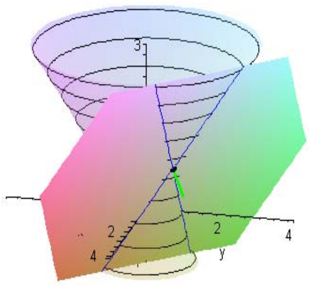
$\tau: F_x(1,1,1) \cdot (x-1) + F_y(1,1,1) \cdot (y-1) + F_z(1,1,1) \cdot (z-1) = 0$

$\Leftrightarrow \quad 2(x-1)+2(y-1)-2(z-1)=0 \quad \Leftrightarrow \quad \underline{x+y-z=1}$

Schnitt $\mathcal{E} \cap H$?

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathcal{E} \cap H \Leftrightarrow \begin{array}{l} x^2+y^2-z^2=1 \\ x+y-z=1 \end{array} \Leftrightarrow \begin{array}{l} x^2+y^2-(x+y-1)^2=1 \\ z=x+y-1 \end{array}$

$\Leftrightarrow \begin{array}{l} \cancel{x^2}+\cancel{y^2}-(\cancel{x^2}+\cancel{y^2}+1+2xy-2x-2y)=1 \\ z=x+y-1 \end{array}$



$\Leftrightarrow \begin{array}{l} xy-x-y+1=0 \\ z=x+y-1 \end{array}$

$\Leftrightarrow \begin{array}{l} (x-1)(y-1)=0 \\ z=x+y-1 \end{array} \Leftrightarrow \begin{array}{l} x=1 \text{ oder } y=1 \\ z=x+y-1 \end{array}$

$\Rightarrow (x=1 \ \underline{\text{und}} \ z=y)$ oder $(y=1 \ \underline{\text{und}} \ z=x)$ $\underline{\text{2 Geraden}}$!